# Introduction to Matrices

In a matrix, numbers are arranged in a box, in rows and columns. In singular, it is called a matrix, while in plural, they are called matrices.

**Rows**: The horizontal set of numbers form the rows
**Columns**: The vertical set of numbers form the columns
**Order of a matrix**: It is the number of rows and columns in a matrix
**Element**: Each number is a matrix is an element

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 5 & 0 \\ 0.6 & -7 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} -4 & 45 \\ 6 & -82 \\ 12 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 4 & 0 & -1 & 6 \end{bmatrix} \qquad E = \begin{bmatrix} 8 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \qquad F = \begin{bmatrix} 6 \end{bmatrix}$$

| **Matrix** | **Rows (m)** | **Columns (n)** | **Order ($m \times n$)** | **Elements** | **Type** |
|---|---|---|---|---|---|
| A | 2 | 2 | $2 \times 2$ | 4 | Square |
| B | 2 | 3 | $2 \times 3$ | 6 | Vertical |
| C | 3 | 2 | $3 \times 2$ | 6 | Horizontal |
| D | 1 | 5 | $1 \times 5$ | 5 | Row matrix |
| E | 5 | 1 | $5 \times 1$ | 5 | Column matrix |
| F | 1 | 1 | $1 \times 1$ | 1 | Singleton matrix |

An element is identified by this notation $Matrix_{row,col}$
$A_{2,1} = 3$
$B_{1,2} = 5$
$C_{3,2} = 0$

**Types of matrices**

| **Type** | **Description** | **m, n** | **Example** |
|---|---|---|---|
| Square matrix | Number of rows and columns are equal | $m = n$ | $\begin{bmatrix} 8 & 2 \\ 6 & 4 \end{bmatrix}$ |
| Horizontal matrix | Columns more than rows | $n > m$ | $\begin{bmatrix} -3 & 0.5 & 0 \\ 0.6 & 7 & -8 \end{bmatrix}$ |
| Vertical matrix | Rows more than columns | $m > n$ | $\begin{bmatrix} 4 & 8 \\ 2 & -1 \\ 9 & 0 \end{bmatrix}$ |
| Row matrix | Only one row | $m = 1$ | $\begin{bmatrix} 2 & 14 & 0 & 1 & -9 \end{bmatrix}$ |
| Column matrix | Only one column | $n = 1$ | $\begin{bmatrix} 8 \\ 9 \\ 0 \\ 4 \\ 1 \end{bmatrix}$ |
| Diagonal matrix | Square. Nonzero elements only in diagonal | $A_{m,m} \neq 0$ $A_{m,n} = 0$ | $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| Null matrix | All elements zero | | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| Unit (Identity) matrix | Diagonal. All elements 1 | $A_{m,m} = 1$ $A_{m,n} = 0$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |

Null matrix can be square or rectangular. But identity matrix must be square.

**Transpose of a matrix**

It means interchanging rows and columns. The number of elements remain the same, but the order changes from $m \times n$ to $n \times m$

| **Matrix** | **Transpose** |
|---|---|
| $A = \begin{bmatrix} 2 & 6 \end{bmatrix}$ | $A^t = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ |
| $A = \begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}$ | $A^t = \begin{bmatrix} 2 & 6 \\ 5 & 9 \end{bmatrix}$ |
| $A = \begin{bmatrix} 9 & 0 & -2 \\ 4 & -1 & 5 \end{bmatrix}$ | $A^t = \begin{bmatrix} 9 & 4 \\ 0 & -1 \\ -2 & 5 \end{bmatrix}$ |
| $A = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$ | $A^t = \begin{bmatrix} 5 & 6 & 8 \end{bmatrix}$ |

| | |
|---|---|
| $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ | $A^t = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ |

**Equality of a matrix**

Matrix A = B if all corresponding elements are equal to each other.

$A = \begin{bmatrix} 4 & 8 & 9 \\ 2 & 7 & 0 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 4 & 8 & 9 \\ 2 & 7 & 0 \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 8 \\ 1 & 0 \end{bmatrix}$

If A=B, then a=3, b=8, c=1, d=0

**Addition and Subtraction of matrices**

To add or subtract, both matrices must have the *same order*.

$A = \begin{bmatrix} 4 & 8 & 9 \\ 2 & 7 & 0 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 6 & -4 & 1 \\ 12 & 10 & -5 \end{bmatrix}$

Order of both is $2 \times 3$

$$A + B = \begin{bmatrix} 4+6 & 8+(-1) & 9+1 \\ 2+12 & 7+10 & 0+(-5) \end{bmatrix} = \begin{bmatrix} 10 & 7 & 10 \\ 14 & 17 & -5 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4-6 & 8-(-1) & 9-1 \\ 2-12 & 7-10 & 0-(-5) \end{bmatrix} = \begin{bmatrix} -2 & 9 & 8 \\ -10 & -3 & 5 \end{bmatrix}$$

**Multiplication of matrix with a scalar**

Scalar means a single number.

$A = \begin{bmatrix} 2 & 5 \\ 6 & 9 \end{bmatrix}$ $\qquad 4A = \begin{bmatrix} 8 & 20 \\ 24 & 36 \end{bmatrix}$

$B = [2 \;\; 14 \;\; 0 \;\; 1 \;\; -9]$ $\qquad -3B = [-6 \;\; -42 \;\; 0 \;\; -3 \;\; 27]$

**Multiplication of a matrix with another matrix**

To multiply two matrices, they must be compatible.

It means that the number of **columns of the first matrix** and the number of **rows of the second** must be same.

$$A_{m,n} \times B_{n,p}$$

A and B matrices are compatible. The resultant product matrix will have the order $m \times p$.

| Matrix A | Matrix B | Compatibility | Product order |
|---|---|---|---|
| $\begin{bmatrix} 2 & 6 \end{bmatrix}_{1,2}$ | $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 7 \end{bmatrix}_{2,3}$ | Yes | $1 \times 3$ |
| $\begin{bmatrix} 3 & 5 & 9 \\ 5 & 2 & 8 \end{bmatrix}_{2,3}$ | $\begin{bmatrix} 9 & 3 & 7 \\ 0 & 3 & 2 \end{bmatrix}_{2,3}$ | No | |
| $\begin{bmatrix} 6 \\ 8 \\ 3 \\ 2 \end{bmatrix}_{4,1}$ | $\begin{bmatrix} 2 & 5 \end{bmatrix}_{1,2}$ | Yes | $4 \times 2$ |
| $\begin{bmatrix} 3 & 5 & 7 \\ 1 & 4 & 9 \\ 0 & 4 & 0 \end{bmatrix}_{3,3}$ | $\begin{bmatrix} 5 & 3 & 1 \\ 6 & 7 & 9 \end{bmatrix}_{2,3}$ | No | |
| $\begin{bmatrix} 4 & 6 & 7 & 8 & 9 \end{bmatrix}_{1,5}$ | $\begin{bmatrix} 6 \\ 8 \\ 3 \\ 5 \\ 9 \end{bmatrix}_{5,1}$ | Yes | $1 \times 1$ |
| $\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}_{3,1}$ | $\begin{bmatrix} 4 \end{bmatrix}_{1,1}$ | Yes | $3 \times 1$ |
| $\begin{bmatrix} 8 & 3 \\ 6 & 0 \end{bmatrix}_{2,2}$ | $\begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}_{2,2}$ | Yes | $2 \times 2$ |
| $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}_{1,3}$ | $\begin{bmatrix} 4 & 7 & 9 \end{bmatrix}_{1,3}$ | No | |
| $\begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}_{3,1}$ | $\begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix}_{3,1}$ | No | |

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \times \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$$

| | |
|---|---|
| $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \times \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$ | $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \times \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$ |
| $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \times \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$ | $\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{bmatrix} \times \begin{bmatrix} b11 & b12 \\ b21 & b22 \\ b31 & b32 \end{bmatrix}$ |

First element of result = sum of the products of the **first row** of A and all elements of the **first column** of B
$c11 = a11 \times b11 + a12 \times b21 + a13 \times b31$

Second element of result = sum of the products of the **first row** of A and all elements of the **second column** of B
$c12 = a11 \times b12 + a12 \times b22 + a13 \times b32$

Once all the columns of the second matrix are completed, start with second row of the first matrix, and repeat as above

Third element of result = sum of the products of the **second row** of A and all elements of the **first column** of B
$c21 = a21 \times b11 + a22 \times b21 + a23 \times b31$

Fourth element of result = sum of the products of the **second row** of A and all elements of the **second column** of B
$c22 = a21 \times b12 + a22 \times b22 + a23 \times b32$

All the rows of first matrix and all the columns of the second matrix are completed.

$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 2 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 6 \\ 5 & 5 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 7 + 4 \times 5 + 3 \times 8 & 2 \times 6 + 4 \times 5 + 3 \times 2 \\ 5 \times 7 + 2 \times 5 + 6 \times 8 & 5 \times 6 + 2 \times 5 + 6 \times 2 \end{bmatrix}$$
$$= \begin{bmatrix} 58 & 38 \\ 93 & 52 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 6 \\ 5 & 5 \\ 8 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 3 \\ 5 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 7 \times 2 + 6 \times 5 & 7 \times 4 + 6 \times 2 & 7 \times 3 + 6 \times 6 \\ 5 \times 2 + 5 \times 5 & 5 \times 4 + 5 \times 2 & 5 \times 3 + 5 \times 6 \\ 8 \times 2 + 2 \times 5 & 8 \times 4 + 2 \times 2 & 8 \times 3 + 2 \times 6 \end{bmatrix}$$
$$= \begin{bmatrix} 44 & 40 & 57 \\ 35 & 30 & 45 \\ 26 & 36 & 36 \end{bmatrix}$$

We can see that $AB \neq BA$
The order is different, and the resultant elements are different.

**Multiplication of matrix with identity matrix**

$$A = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AI = \begin{bmatrix} 2 \times 1 + 0 & 0 - 4 \times 1 \\ 5 \times 1 + 0 & 0 + 6 \times 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 5 & 6 \end{bmatrix}$$

$$AI = A$$

Just as $5 \times 1 = 5$, multiplying any matrix with identity matrix gives the same matrix.

Notes to remember:
a. $AB \neq BA$
b. $(A+B)(A-B) \neq A^2 - B^2$
c. $A + B = B + A$
d. $AI = A$
e. $A(B+C) \neq AB + AC$

**Applications of matrices**
* Matrices are used to solve linear equations.

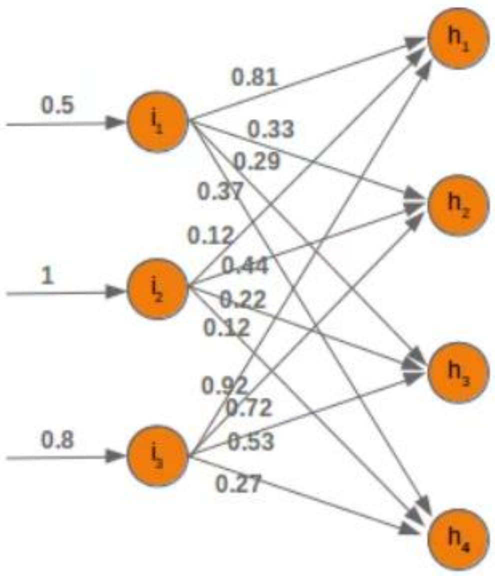
$$32a + 62b - c + 74d = 36$$
$$12a - 22b - 52c + 18d = 78$$
$$50a + 42b + 29c - 83d = 107$$
$$27a - 59b + 92c - 47d = 123$$

It is too complex to solve linear equations when unknowns are more than two. The values are written in matrix form.

$$\begin{bmatrix} 32 & 62 & -1 & 74 \\ 12 & -22 & -52 & 18 \\ 50 & 42 & 29 & -83 \\ 27 & -59 & 92 & -47 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 36 \\ 78 \\ 107 \\ 123 \end{bmatrix}$$

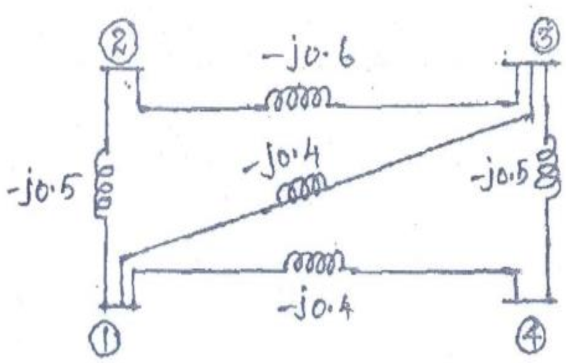
With suitable operations, the values of the unknowns are calculated.

* They have applications in engineering, artificial intelligence, data processing, computer graphics, optics, etc.




$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \\ w_{41} & w_{42} & w_{43} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

* In electrical engineering, electric current flowing through various power lines can be calculated easily by placing the known values of voltage and admittances in matrix form.




$$Y_{BUS} = \begin{bmatrix} -j1.3 & j0.5 & j0.4 & j0.4 \\ j0.5 & -j1.1 & j0.6 & 0 \\ j0.4 & j0.6 & -j1.5 & j0.5 \\ j0.4 & 0 & j0.5 & -j0.9 \end{bmatrix}$$